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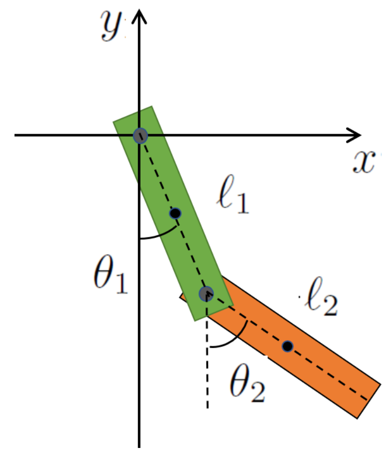
Equations of Motion for Double Pendulum

We consider the double pendulum, which consists of the arm 1 and the arm 2 (See Figure). The lengths of each arm are ℓ_1 and ℓ_2, and the masses are m_1 and m_2. Suppose that the centers of mass (COM) of each arm are at the centers of the each arm, and the inertia moments of each arm are L_1 and L_2, respectively.



The configuration of the double pendulum is specified by the angles between each arm and the negative direction of the y axis, θ_1 and θ_2.

Using the formalism of analytical mechanics, we will derive the equations of motion for the system as a set of ordinary differential equaions of the first order with 4 variables, i.e. the angles (θ_1, θ_2) and their canonical conjugate momenta (p_1, p_2).

First, we derive the Lagrangian L. Suppose that the co-ordinates of COM of each arm be (x_1, y_1) and (x_2, y_2), then they can be expressed in terms of the angles (θ_1, θ_2) as

$$\begin{cases} x_1 = & \frac{1}{2}\ell_1 \sin\theta_1, \\ y_1 = & -\frac{1}{2}\ell_1 \cos\theta_1, \end{cases} \qquad \begin{cases} x_2 = & \ell_1 \sin\theta_1 + \frac{1}{2}\ell_2 \sin\theta_2, \\ y_2 = & -\ell_1 \cos\theta_1 - \frac{1}{2}\ell_2 \cos\theta_2. \end{cases}$$

By taking the time derivatives of these equations, the velocities of each arms $(\dot{x}_1, \dot{y}_1)$ and $(\dot{x}_2, \dot{y}_2)$ are given by

$$\begin{cases} \dot{x}_1 = & \frac{1}{2}\ell_1 \cos\theta_1 \dot{\theta}_1, \\ \dot{y}_1 = & \frac{1}{2}\ell_1 \sin\theta_1 \dot{\theta}_1, \end{cases} \qquad \begin{cases} \dot{x}_2 = & \ell_1 \cos\theta_1 \dot{\theta}_1 + \frac{1}{2}\ell_2 \cos\theta_2 \dot{\theta}_2, \\ \dot{y}_2 = & \ell_1 \sin\theta_1 \dot{\theta}_1 + \frac{1}{2}\ell_2 \sin\theta_2 \dot{\theta}_2 \,. \end{cases}$$

The kinetic energy T may be expressed as the sum of the translational energy of COM and the rotational energy around COM, thus is given by

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}L_1\dot{\theta}_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}L_2\dot{\theta}_2^2 \\ &= \frac{1}{2}(\dot{\theta}_1, \dot{\theta}_2) \begin{pmatrix} m_1\left(\frac{1}{2}\ell_1\right)^2 + L_1 + m_2\ell_1^2, & \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_1-\theta_2) \\ \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_1-\theta_2), & m_2\left(\frac{1}{2}\ell_2\right)^2 + L_2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \equiv \frac{1}{2}\dot{\boldsymbol{\theta}}^t \hat{A} \dot{\boldsymbol{\theta}} \end{aligned}$$

as a quadratic form of $\dot{\boldsymbol{\theta}} \equiv (\dot{\theta}_1, \dot{\theta}_2)$, using the symmetric matrix $\hat{A}$.

The potential energy U is given by

$$U = -m_1 g \frac{1}{2}\ell_1 \cos\theta_1 - m_2 g \left(\ell_1\cos\theta_1 + \frac{1}{2}\ell_2\cos\theta_2\right),$$

thus we have Lagrangian L for this system

$$L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = T(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - U(\boldsymbol{\theta})$$

as a function of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$.

The momentum $\boldsymbol{p} \equiv (p_1, p_2)$ canonical conjugate to $\boldsymbol{\theta}$ is defined as

$$\boldsymbol{p} \equiv \frac{\partial L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \dot{\boldsymbol{\theta}}} = \hat{A}\dot{\boldsymbol{\theta}}, \tag{1}$$

thus the Lagrange equation of motion is given as

$$\dot{\boldsymbol{p}} = \frac{\partial L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}, \tag{2}$$

or

$$\begin{cases} \dot{p}_1 &= \dfrac{\partial L}{\partial \theta_1} = -\dfrac{1}{2} m_2 \ell_1 \ell_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - m_1 g \dfrac{1}{2}\ell_1 \sin\theta_1 - m_2 g \ell_1 \sin\theta_1 \,, \\ \dot{p}_2 &= \dfrac{\partial L}{\partial \theta_2} = +\dfrac{1}{2} m_2 \ell_1 \ell_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2 g \dfrac{1}{2}\ell_2 \sin\theta_2 \,. \end{cases}$$

The equations of motion for the system are given by Eqs. (1) and (2), which are the second order differential equations of $\boldsymbol{\theta}$. In the numerics, it is more convenient in the form

$$\boxed{\dot{\boldsymbol{\theta}} = \hat{A}^{-1}\boldsymbol{p}, \qquad \dot{\boldsymbol{p}} = \frac{\partial L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}\bigg|_{\dot{\boldsymbol{\theta}} = \hat{A}^{-1}\boldsymbol{p}}} \tag{3}$$

as first order differential equations of the four variables $\boldsymbol{\theta}$ and $\boldsymbol{p}$, which we solve numerically using the Runge-Kutta method of the fourth order. Note that $\hat{A}^{-1}$ is the inverse matrix of $\hat{A}$, namely

$$\hat{A}^{-1} = \frac{1}{\det}\begin{pmatrix} b, & -c \\ -c, & a \end{pmatrix} \quad \text{for symmetric matrix} \quad \hat{A} = \begin{pmatrix} a, & c \\ c, & b \end{pmatrix}, \quad \det \equiv ab - c^2 \,.$$

Note that this set of equations (3) is exactly the same as Hamilton canonical equation

$$\dot{\boldsymbol{\theta}} = \frac{\partial H(\boldsymbol{\theta}, \boldsymbol{p})}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = -\frac{\partial H(\boldsymbol{\theta}, \boldsymbol{p})}{\partial \boldsymbol{\theta}}; \qquad H(\boldsymbol{\theta}, \boldsymbol{p}) \equiv \frac{1}{2}\boldsymbol{p}^t \hat{A}^{-1}\boldsymbol{p} + U(\boldsymbol{\theta}) \,.$$